Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $a,b > 0$. Prove that for any x,y holds inequality

$|a\cos x + b\cos y| \le \sqrt{a^2 + b^2 + 2ab\cos(x+y)}$

and find when equality occurs.

Solution.

Indeed, $|a\cos x + b\cos y| \le \sqrt{a^2 + b^2 + 2ab\cos(x+y)} \iff$

$(a\cos x + b\cos y)^2 \le a^2 + b^2 + 2ab\cos(x+y) \iff$

$a^2\cos^2 x + b^2\cos^2 y + 2ab\cos x\cos y \le a^2 + b^2 + 2ab\cos(x+y) \iff$

$0 \le a^2\sin^2 x + b^2\sin^2 y + 2ab(\cos(x+y) - \cos x\cos y) \iff$

$0 \le (a\sin x - b\sin y)^2$.

Equality occurs iff $a\sin x - b\sin y = 0 \iff a\sin x = b\sin y$. Let $\varphi := x+y$, then

$\sin x \div \sin y = \dfrac{1}{a} \div \dfrac{1}{b} = bc \div ca \iff \sin x = kbc, \sin y = kca$ and

$\sin y = \sin(\varphi - x) \iff \sin y = \sin\varphi\cos x - \cos\varphi\sin x \iff$

$kca = \sin\varphi\sqrt{1 - k^2b^2c^2} - kbc\cos\varphi \iff (kca + kbc\cos\varphi)^2 = \sin^2\varphi - k^2b^2c^2\sin^2\varphi \iff$

$k^2c^2a^2 + k^2b^2c^2\cos^2\varphi + 2abc^2\cos\varphi = \sin^2\varphi - k^2b^2c^2\sin^2\varphi \iff$

$k^2c^2(a^2 + b^2 + 2ab\cos\varphi) = \sin^2\varphi \iff k^2 = \dfrac{\sin^2\varphi}{c^2(a^2 + b^2 + 2ab\cos\varphi)}$.

Hence, $\cos^2 x = 1 - k^2b^2c^2 = 1 - \dfrac{b^2\sin^2\varphi}{a^2 + b^2 + 2ab\cos\varphi} = \dfrac{(a + b\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$

and $\cos^2 y = 1 - k^2c^2a^2 = \dfrac{(b + a\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$.

(Obviously that $\dfrac{(a + b\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi} \le 1$ and $\dfrac{(b + a\cos\varphi)^2}{a^2 + b^2 + 2ab\cos\varphi}$).